In the case of this example it is easy to show that the (x, y) discontinuity cannot be greater than or equal to k (where k is some definite positive number) at every point of the set. From theorem 1 of section 7 it follows that if a function of x and y has an (x, y) discontinuity greater than or equal to k at a set of points that is everywhere dense, then it must have such an (x, y) discontinuity also at the limiting points of the set. The function $f(x, y, 0)$ of the example is a function that has an (x, y) discontinuity at a set of points that is everywhere dense. Hence if this discontinuity were greater than or equal to k at every point of the set, $f(x, y, 0)$ would have an (x, y) discontinuity greater than or equal to k at every point of the region which is impossible.

We now wish to see if the function $f(x, y, z_0)$ can be discontinuous in (x, y) together at every point of the region. It can be proved that this cannot be the case, but before doing so we need a lemma concerning a function of two variables.

*Lemma.**—If the function $f(x, y)$ is discontinuous in (x, y) together at every point (x, y) of a region, then there always exists a sub-region within which the (x, y) discontinuity is greater than a sufficiently small quantity $A > 0$ at every point.

In order to prove this lemma we must show that the points at which $\lim\limits_{\rho = 0} k < \sigma$, where $\sigma > 0$, cannot be everywhere dense in any sub-region, where k represents the oscillation of the function within any circle of radius ρ. Suppose that these points are everywhere dense in some sub-region. Now choose a set of positive quantities

$$\sigma_1 > \sigma_2 > \sigma_3 > \ldots\ldots\ldots > \sigma_n > \ldots.$$

where $\lim\limits_{n = \infty} \sigma_n = 0$. Since we have assumed that the points at which $\lim\limits_{\rho = 0} k < \sigma$ are everywhere dense in every sub-region, we can within any sub-region find a circle R_0 of radius $r_0 > 0$ about some point (x_0, y_0) such that at every point within this circle the oscillation k of the function is less than σ_1. Hence there exists a circle R_1 of radius r_1 greater than zero and center at (x_1, y_1) and lying wholly within the circle R_0, such that at every point within this circle R_1 the oscillation k is less than σ_2. In a like manner we can determine a circle R_2

* See E. J. Townsend: Ueber den Begriff und die Anwendung des Doppellimes, 1900.

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of radius $r_2 > 0$ and center (x_2, y_2) and lying wholly within the circle R_1, such that the oscillation k is less than σ_3 at every point within the circle R_2. Continue this process and we obtain in general a circle R_n of radius r_n greater than zero and center at (x_n, y_n) and lying entirely within the circle R_{n-1}, such that the oscillation k of the function is less than σ_n at every point within the circle R_n. Since now $r_1 > r_2 > r_3 > \ldots\ldots\ldots > r_n > \ldots$, we have $\lim_{n=\infty} r_n = 0$, while at the same time the centers of the circles approach a definite limiting point (x_0, y_0). At this point therefore we have $\lim_{\rho=0} k = 0$, where ρ is the radius of any circle about (x_0, y_0). Hence the function $f(x, y)$ is continuous in (x, y) together at this point, which contradicts the hypothesis. The lemma is therefore true.

Theorem 8.*—If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$\alpha < x < \beta, \quad a < y < b, \quad z_0 < z < z_n,$$

and if at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

we have $\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$, then $f(x, y, z_0)$ cannot be discontinuous in (x, y) together at every point of the region

$$\alpha < x < \beta, \quad a < y < b, \tag{B}$$

or of any sub-region thereof.

Suppose that the limiting function $f(x, y, z_0)$ could have an (x, y) discontinuity at every point of region (B). Then there must exist, on $z = z_0$, a circle of radius $r_0 > 0$, such that at every point of this circle the (x, y) discontinuity of the function is greater than a sufficiently small quantity $A > 0$. This follows from the lemma we have just proved. Let (x_0, y_0) be the center of the above circle. Since by hypothesis $\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$ at every point of the region (A) we can choose an arbitrarily small $\sigma > 0$ and can then find a $z = z_1$, such that

$$|f(x_0, y_0, z_1) - f(x_0, y_0, z_0)| < 1/2\sigma \tag{1}$$

where

$$z_0 < z_1 \leqq z_0 + \theta_0, \quad \theta_0 > 0.$$

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*Baire gives the corresponding theorem for a function of a single variable. See Annali di Matematica, 1899.

But the function $f(x, y, z)$ is by hypothesis continuous in (x, y) together throughout region (B) and for every z for which $f(x, y, z)$ is defined, and hence there must exist for the same σ a circle of radius r_1 about (x_0, y_0) in the plane $z = z_1$, such that for every point within this circle

$$|f(x, y, z_1) - f(x_0, y_0, z_1)| < 1/2\,\sigma \tag{2}$$

where $$z_0 < z_1 \leqq z_0 + \theta_0.$$

Let $r_1 < r_0$ so that inequalities (1) and (2) hold for every point (x, y) within this circle and we may therefore add these inequalities and thus obtain

$$|f(x, y, z_1) - f(x_0, y_0, z_0)| < \sigma. \tag{3}$$

This holds for every point (x, y) within the circle of radius r_1 about (x_0, y_0). Since by assumption $f(x, y, z_0)$ is to be discontinuous in (x, y) together at every point of region (B), there must exist in the plane $z = z_0$ within the circle of radius r_1 about (x_0, y_0) at least one point (x_1, y_1), such that

$$|f(x_1, y_1, z_0) - f(x_0, y_0, z_0)| > A. \tag{4}$$

But by hypothesis $\lim\limits_{z = z_0} f(x_1, y_1, z) = f(x_1, y_1, z_0)$ and hence there must exist a plane $z = z_2$, such that for the same σ as above

$$|f(x_1, y_1, z_2) - f(x_1, y_1, z_0)| > 1/2\,\sigma, \tag{5}$$

where $$z_0 < z_3 < z_1.$$

Since $f(x, y, z_2)$ is continuous in (x, y) together throughout region (B), there must exist for the same σ in the plane $z = z_2$ a circle of radius r_2 about (x_1, y_1), lying entirely within the circle of radius r_1, such that for every point (x, y) within this circle

$$|f(x, y, z_2) - f(x_1, y_1, z_2)| < 1/2\,\sigma \tag{6}$$

where $$z_0 < z_2 < z_1.$$

Let $r_2 < r_1$, so that inequalities (5) and (6) hold for every point (x, y) within the circle of radius r_2. We may therefore add these inequalities and thus obtain

$$|f(x, y, z_2) - f(x_1, y_1, z_0)| < \sigma \tag{7}$$

for every point (x, y) within the circle of radius r_2. We have moreover assumed that $f(x, y, z_0)$ is discontinuous in (x, y) together at every point of region (B) and hence there must exist in the plane $z = z_0$ at least one point (x_2, y_2) within the circle of radius r_2 about (x_1, y_1) for which

$$|f(x_2, y_2, z_0) - f(x_1, y_1, z_0)| > A. \tag{8}$$

We can continue this reasoning indefinitely and have, in general, that there exists a circle of radius r_n about (x_{n-1}, y_{n-1}) and lying wholly within each of the preceding circles and that there also exists a corresponding plane $z = z_n$, where $z_1 > z_2 > z_3 > \ldots > z_n > \ldots > z_0$, such that for every point (x, y) within this circle the following two inequalities hold

$$|f(x, y, z_n) - f(x_{n-1}, y_{n-1}, z_0)| < \sigma \tag{9}$$

$$|f(x_{n-1}, y_{n-1}, z_0) - f(x_{n-2}, y_{n-2}, z_0)| > A. \tag{10}$$

Now as n increases indefinitely, the radius r_n decreases indefinitely and approaches zero as limit. The centers (x_n, y_n) of the circles must at the same time approach a definite fixed limiting point $(\bar{x}, \bar{y})$. We thus obtain the following inequalities

$$|f(x_0, y_0, z_0) - f(\bar{x}, \bar{y}, z_1)| < \sigma \qquad \text{(from 3)}$$

$$|f(\bar{x}, \bar{y}, z_2) - f(x_1, y_1, z_0)| < \sigma \qquad \text{(from 7)}$$

$$|f(x_1, y_1, z_0) - f(x_0, y_0, z_0)| > A.$$

If now we choose $0 < \sigma < 1/4\,A$ and combine these three inequalities we obtain

$$|f(\bar{x}, \bar{y}, z_2) - f(\bar{x}, \bar{y}, z_1)| > 1/2\,A.$$

In a similar manner we obtain the following inequalities

$$|f(\bar{x}, \bar{y}, z_3) - f(\bar{x}, \bar{y}, z_2)| > 1/2\,A$$

$$|f(\bar{x}, \bar{y}, z_4) - f(\bar{x}, \bar{y}, z_3)| > 1/2\,A$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|f(\bar{x}, \bar{y}, z_n) - f(\bar{x}, \bar{y}, z_{n-1})| > 1/2\,A.$$

But the z's are dense at $z = z_0$ and hence we obtain from the last inequality

$$\lim_{\substack{z_{n-1} = z_0 \\ z_n = z_0}} [f(\bar{x}, \bar{y}, z_n) - f(\bar{x}, \bar{y}, z_{n-1})] > 1/2\,A \neq 0.$$

and hence the limit $\lim\limits_{z = z_0} f(x, y, z)$ does not exist at the point $(\bar{x}, \bar{y})$. The assumption that $f(x, y, z_0)$ has an (x, y) discontinuity at every point of region (B) is therefore false and the theorem is true.

Remark 1.—In the above theorem we assumed that $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every point of region (A), but we were only able to prove that the

limiting function $f(x, y, z_0)$ is not discontinuous in (x, y) together at every point of region (B). As a matter of fact, $f(x, y, z_0)$ may have an (x, y) discontinuity at every point of the boundaries of region (A). We show this by means of the following example.

Example 2. Let $f(x, y, z) = \dfrac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1}$,

where $0 \leqq x \leqq 10$, $0 \leqq y \leqq 10$, $0 < z \leqq 10$, and $\lim\limits_{z=0} f(x, y, z) = f(x, y, 0)$.

The value of the limit $\lim\limits_{z=0} f(x, y, z)$ is evidently zero for all values of x and y which make xy integral, for in this case $\sin \pi xy = 0$. But if we let $2n < xy < 2n + 1$, where n is some integer, then for all such points we have $0 < \sin \pi xy < 1$. Hence for all such values of x and y

$$\lim_{z=0} \frac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1} = \lim_{z=0} \frac{1 - \dfrac{1}{(1 + \sin \pi xy)^{1/z}}}{1 + \dfrac{1}{(1 + \sin \pi xy)^{1/z}}} = +1 .$$

On the other hand if we let $2n + 1 < xy < 2n + 2$, where n is a positive integer, then we obtain

$$\lim_{z=0} \frac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1} = \frac{0 - 1}{0 + 1} = -1 .$$

But when either x or y is equal to zero, then $f(x, y, 0) = 0$ along the whole X-axis and the whole Y-axis. Hence $f(x, y, 0)$ is discontinuous in (x, y) together at every point of these lines.

Remark 2.—In theorem 7 we showed that, for the example there given, the (x, y) discontinuity could not be greater than or equal to k at every point. We are now in a position to prove in general that, if the function $f(x, y, z)$ satisfies the conditions of theorem 7, then the limiting function $f(x, y, z_0)$ cannot have an (x, y) discontinuity greater than or equal to k at a set of points that is everywhere dense. For if this were possible, then it follows from theorem 1 of section 7 that the function $f(x, y, z_0)$ has an (x, y) discontinuity greater than or equal to k also at the limiting points, that is to say it would have such an (x, y) discontinuity at every point of the region. This is however impossible by theorem 8 just proved.

Theorem 9.—If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha < x < \beta, \quad y_0 < y < y_m, \quad z_0 < z < z_n,$$

and if for every point of the interval $\alpha \leqq x \leqq \beta$, $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$, then $f(x, y_0, z_0)$ cannot be discontinuous in x alone at every point of the interval $\alpha < x < \beta$.

The proof of this theorem is strictly analogous to that of theorem 8. We replace the circles by intervals about x_0 on lines parallel to the X-axis, in a manner similar to that employed in theorem 6.

Theorem 10.—If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$\alpha < x < \beta, \quad a < y < b, \quad z_0 < z < z_n,$$

and if at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

$\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$, then the function $f(x, y, z_0)$ must be continuous in (x, y) together at a set of points that is at least dense everywhere in the region

$$\alpha < x < \beta, \quad a < y < b. \tag{B}$$

Suppose that this is not so. Then there must exist in the plane $z = z_0$ some circle R_0 of radius r_0 greater than zero and lying entirely within region (B), such that within this circle there is no point for which $\lim\limits_{\rho = 0} k = 0$, where ρ is the radius of any variable circle about the point and k is the oscillation of the function within this circle. Within the circle R_0 therefore $\lim\limits_{\rho = 0} k \neq 0$ for every point. Hence every point (x, y) within the circle R_0 is a point at which $f(x, y, z_0)$ has an (x, y) discontinuity. This however contradicts theorem 8, and hence the theorem is true.

This theorem may be extended in a manner similar to that employed in the extension of theorem 5 of section 7. We thus obtain the following corollary.

Corollary.—If the function $f(x, y, z)$ satisfies the conditions of theorem 10, then the points at which $f(x, y, z_0)$ is continuous in (x, y) together form a set of the second category, and hence a non-enumerable set.

Theorem 11.—If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha < x < \beta, \quad y_0 < y < y_m, \quad z_0 < z < z_n,$$

and if $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every point of the interval $\alpha \leqq x \leqq \beta$, then $f(x, y_0, z_0)$ is continuous in x alone at a set of points that is at least everywhere dense in the interval $\alpha < x < \beta$. Moreover this set belongs to the second category and is non-enumerable.

Suppose that this is not true. Then there must exist on the X-axis some interval greater than zero and lying wholly within the interval $\alpha < x < \beta$, such that within this interval there is no point x for which the limit of the oscillation of the function $f(x, y_0, z_0)$ is zero. Within this interval therefore the function $f(x, y_0, z_0)$ has an x discontinuity at every point. But this contradicts theorem 9, and hence our theorem is true as regards the first part.

The second part of the theorem is proved in exactly the same manner as was the corollary of the preceding theorem.

CHAPTER III.

SERIES OF FUNCTIONS OF TWO REAL VARIABLES.

10. *Properties of Series of Functions of Two Real Variables.*—Suppose that we have given

$$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y),$$

where each u is finite and a continuous function of (x, y) together throughout a certain region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b,$$

for every value of n $(n = 1, 2, 3, \dots)$. Since each u is finite and continuous in (x, y) together and since n is also finite, it follows that $S_n(x, y)$ must be finite and continuous in (x, y) together throughout the given region. If then for every

constant value of x and y of the region, $\lim_{n=\infty} S_n(x, y) = S(x, y)$, then $S(x, y)$ represents a function of (x, y) throughout the given region, and the question now arises as to the properties of this limiting function $S(x, y)$.

If we wish, we may consider $S_n(x, y)$ also as a function of the three variables x, y, and n. This enables us to apply the results of the preceding chapters to the present problem. In order to do so, we define x and y as above, namely as any values of

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b.$$

In order to define z we put $z = 1/n$, so that the z of the preceding chapters takes on the values $z = 1, 1/2, 1/3, \ldots, 1/n \ldots$, which is a set of values dense at $z = 0$. We then have

$$S(x, y) = \lim_{n=\infty} S_n(x, y) = \lim_{z=0} f(x, y, z),$$

where $f(x, y, z) = S_n(x, y)$ for every point (x, y) of the given region. It follows from this that everything which was shown to be true for the limiting function $f(x, y, z_0)$ in the case of a function of three variables must also hold for the series as defined above. The converse of this is, however, not true, for if we have proved a theorem about an infinite series, it does not follow necessarily that an analogous theorem exists for a function of three variables. The reason for this is to be found in the fact that, in the case of a function of three variables, z is defined for all positive values, while in the case of the series it is defined only for a set of values dense at $z = 0$. It may therefore happen that a theorem holds for a set of values dense at $z = z_0$, where $z_0 < z \leqq z_0 + \theta_0$, while the theorem will not hold for *all* values z of the interval $z_0 < z \leqq z_0 + \theta_0$.

From what has been said follows that to the three different kinds of uniform convergence of $f(x, y, z)$ to its limiting function, as they were defined in the last chapter, there correspond three different kinds of uniform convergence of $S_n(x, y)$ to $S(x, y)$. These are as follows:

(1) If $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$ is, for every finite value of n, finite and continuous in (x, y) together throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if for every point (x, y) of this region $\lim_{n=\infty} S_n(x, y) = S(x, y)$, then $S_n(x, y)$

converges uniformly to $S(x, y)$ throughout region (A), if having chosen an arbitrarily small $\sigma > 0$, we can then find a positive integer m, such that

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma,$$

when $n > m$ and for all points (x, y) of region (A).

(2) If $S_n(x, y)$ and $S(x, y)$ are defined as in (1), then $S_n(x, y)$ converges "einfach gleichmässig" throughout region (A) to $S(x, y)$, if having chosen an arbitrarily small $\sigma > 0$ we can then find a positive integer m, such that

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma,$$

for some set of integral values n greater than m, which however increase without limit. In this case the n's do not assume all integral values.

(3) If $S_n(x, y)$ and $S(x, y)$ are defined as in (1), then $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout the region (A), if having chosen an arbitrarily small $\sigma > 0$, there then exists a plane $z = 1/n$ and between this plane $z = 1/n$ and the plane $z = 0$ a finite number of planes, and on each of these a finite number of rectangles, having their sides parallel to the coordinate axes, such that their projections on the plane $z = 0$ completely fill the region (A) and that for every point (x, y) of these rectangles

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma.$$

We will now give the theorems that deal with the properties of a series of functions of two real variables. Unless something is said to the contrary, we shall in every case make the following assumptions about the series:

(1) $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y),$

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

where $u_i(x, y)$, $(i = 1, 2, 3, \dots, n)$ is continuous in (x, y) together throughout region (A);

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, for every constant pair of values (x, y) of region (A).

From theorem 1 of section 8 we have immediately

Theorem 1.—The necessary and sufficient condition that $S_n(x, y)$ shall converge uniformly to $S(x, y)$ throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

is that at every point (x_0, y_0) of this region, boundaries included, the simultaneous

limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists and equals $S(x_0, y_0)$.

Example 1.

$$\text{Let } S_n(x, y) = \frac{xy}{xy+1} + \left[\frac{2xy}{4xy+1} - \frac{xy}{xy+1}\right] + \cdots\cdot + \left[\frac{nxy}{n^2xy+1} - \frac{(n-1)xy}{(n-1)^2xy+1}\right] = \frac{nxy}{n^2xy+1}$$

where $$0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1.$$

In this case the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists and equals zero at every point (x, y) of the region, and hence $S_n(x, y)$ converges uniformly to $S(x, y)$ throughout the region.

Example 2.

$$\text{Let } S_n(x, y) = \frac{xy}{x^3+y^3+1} + \left[\frac{4xy}{8x^3+8y^3+1} - \frac{xy}{x^3+y^3+1}\right] + \cdots\cdot + \left[\frac{n^2xy}{n^3x^3+n^3y^3+1} - \frac{(n-1)^2xy}{(n-1)^3x^3+(n-1)^3y^3+1}\right] = \frac{n^2xy}{n^3x^3+n^2y^3+1}$$

where $$0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1.$$

Here the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ does not exist for $x = y = 0$ and hence $S_n(x, y)$ does not converge uniformly to $S(x, y)$.

From theorem 1 of section 9 follows:

Theorem 2.—If $S_n(x, y)$ converges uniformly to $S(x, y)$ throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b,$$

then $S(x, y)$ is continuous in (x, y) together throughout the given region.

This condition is only a sufficient condition for the continuity of $S(x, y)$ in (x, y) together. Thus in example 2 the function $S(x, y)$ is continuous in (x, y) together throughout the region given, but $S_n(x, y)$ does not converge uniformly to $S(x, y)$.

The necessary and sufficient condition for the continuity of $S(x, y)$ in (x, y) together at some fixed point (x_0, y_0) of the region follows from theorem 3 of section 9. It may be stated as follows:

Theorem 3.—The necessary and sufficient condition that the function $S(x, y)$ is continuous in (x, y) together at a fixed point (x_0, y_0) of the given region, is that having chosen an arbitrarily small $\sigma > 0$ and having some set of positive integers $m_1, m_2, m_3, \ldots, m_k, \ldots$, which increase indefinitely, there shall then exist about (x_0, y_0) corresponding to each m_k a circle of radius $r_k > 0$ (which may vary with m_k), such that for all points (x, y) within this circle

$$|S_{m_k}(x, y) - S(x, y)| < \sigma.$$

From theorem 5 of section 9 we obtain the necessary and sufficient condition that $S(x, y)$ is continuous in (x, y) together throughout the given region. The condition may be stated as follows:

Theorem 4.—The necessary and sufficient condition that $S(x, y)$ is continuous in (x, y) together at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

is that for every arbitrarily small $\sigma > 0$ there shall exist a plane $z = 1/n$ and between this plane and the plane $z = 0$ a finite set of planes parallel to the XY-plane, each of these to contain a finite set of rectangles whose sides are parallel to the coordinate axes and whose projections on the plane $z = 0$ completely fill the region (A), such that for every point (x, y) of these rectangles

$$|S_n(x, y) - S(x, y)| < \sigma.$$

Theorem 5.—If the sum of the maximum values of the terms of a series $S_n(x, y)$ converges absolutely within a given region, then the series converges also uniformly.

The following conditions are given:

(1) $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

Here as elsewhere we confine ourselves to the case where $u_i\,(x, y)$ is continuous in (x, y) together;

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every pair of values (x, y) of region (A);

(3) $|u_i(x, y)| = M_i$, where M_i is positive and finite, and this is to hold for every (x, y) of region (A);

(4) $\sum\limits_{i=1}^{\infty} M_i$ converges.

We are to show that under these conditions $S_n(x, y)$ converges uniformly throughout region (A).

Since $\sum\limits_{i=1}^{\infty} M_i$ is convergent, we can choose an arbitrarily small $\varepsilon > 0$ and can then find an m, such that

$$(M_{m+1} + M_{m+2} + \ldots + M_{m+p}) < \varepsilon, \; p = 1, 2, 3, \ldots .$$

Hence for m sufficiently large

$$|u_{m+1}(x, y) + u_{m+2}(x, y) + \ldots + u_{m+p}(x, y)| < \varepsilon, \; p = 1, 2, \ldots$$

for every point (x, y) of region (A). We therefore have at every point of this region

$$|G - S_n(x, y)| < \varepsilon, \text{ where } n > m,$$

and hence the simultaneous limit $\lim\limits_{\substack{x=x_0\\y=y_0\\n=\infty}} S_n(x, y)$ exists at every point of region (A), boundaries included.

From 4 of the hypothesis follows that $S_n(x, y)$ also converges and hence by virtue of theorem 3 of section 2

$$\lim_{\substack{x=x_0\\y=y_0\\n=\infty}} S_n(x, y) = \lim_{n=\infty} \lim_{\substack{x=x_0\\y=y_0}} S_n(x, y).$$

But $S_n(x, y)$ is continuous in (x, y) together and hence

$$\lim_{n=\infty} \lim_{\substack{x=x_0\\y=y_0}} S_n(x, y) = \lim_{n=\infty} S_n(x, y) = S(x_0, y_0)$$

at every point (x_0, y_0) of region (A). The simultaneous limit therefore not only

exists, but it also equals the value of the function at the point, and hence the series converges uniformly by theorem 1 of this section.

In theorem 3 we have given the necessary and sufficient condition that the limiting function $S(x, y)$ is continuous in (x, y) together. The question now presents itself as to whether the limiting function may be discontinuous and, if it is discontinuous, how often it may be discontinuous. We obtain an answer to this question from theorem 7 of section 9. It is as follows:

Theorem 6.—If $S_n(x, y)$ and $S(x, y)$ are defined as in theorem 1, then the limiting function $S(x, y)$ may be discontinuous in (x, y) together at a set of points that is everywhere dense in the given region, but in this case the (x, y) discontinuities cannot all be greater than or equal to some finite positive number $\bar{k}$.

The proof follows immediately from the example under theorem 7 of section 9. If in this we let $z = 1/n$, then the example becomes

$$S_n(x, y) = \sum_{k=1}^{n} 1/k! \left[(\cos k!\,\pi x)^{2n} + (\cos k!\,\pi y)^{2n} \right],$$
$$0 \leqq x < 1, \quad 0 \leqq y < 1.$$

From theorem 8 of section 9 we obtain

Theorem 7.—If $S_n(x, y)$ and $S(x, y)$ are defined as before, then the limiting function $S(x, y)$ cannot be discontinuous in (x, y) together at every point of the region $\alpha < x < \beta$, $a < y < b$.

From theorem 10 of section 9 we have

Theorem 8.—If $S_n(x, y)$ and $S(x, y)$ are defined as before, then $S(x, y)$ must be continuous in (x, y) together at a set of points that is at least everywhere dense in the region $\alpha < x < \beta$, $a < y < b$. Moreover this set of points is a set of the second category and hence is non-enumerable.

11. *The Integration of Series Term by Term.*

In this section we shall consider the following question. Suppose we know:

(1) that $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$, and that the simultaneous integral $\int_\alpha^\beta \int_a^b u_k(x, y)\,dxdy$ exists for every finite value of k. Then the simultaneous integral $\int_\alpha^\beta \int_a^b S_k(x, y)\,dxdy$ also exists for every finite value of k, where our region is defined as

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

(2) We have also $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every constant pair of values (x, y) of region (A).

This being the case the question arises under what conditions will the integral of $S(x, y)$ exist over the region, and if it exists, under what conditions will it equal the term by term integral? This is to say, under what conditions do we have

$$\int_\alpha^\beta \int_a^b \lim_{n=\infty} S_n(x, y)\, dxdy = \lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy?$$

Before answering these questions, let us first explain what we mean by the simultaneous integral. In dealing with a function of two variables we meet with two kinds of integrals, the twice-taken or double integral and the simultaneous integral.

The twice-taken or double integral is defined as the result of integrating the function $f(x, y)$ first with respect to one of the variables, say x, between the given limits (α, β), and then integrating the resulting function with respect to y between the limits (a, b). This gives us the integral which is denoted by $\int_a^b dy \int_\alpha^\beta f(x, y)\, dx$. The integral in question is equivalent to a double limit.

The simultaneous integral $\int_\alpha^\beta \int_a^b f(x, y)\, dxdy$ may be defined as follows: Divide the region into a set of rectangles by means of lines parallel to the X and Y axes. Then multiply the area of each rectangle by the value of the function at some point of the corresponding rectangle, and add the results. We then have

$$\begin{aligned}\Sigma = (x_0 - \alpha)(y_0 - a) f(\bar{x}_0, \bar{y}_0) + (x_1 - x_0)(y_0 - a) f(\bar{x}_1, \bar{y}_0) + \ldots\ldots\ldots \\ + (x_i - x_{i-1})(y_k - y_{k-1}) f(\bar{x}_i, \bar{y}_k) + \ldots\ldots + (\beta - x_{l-1})(b - y_{m-1}) f(\bar{x}_l, \bar{y}_m)\end{aligned}$$

where $f(\bar{x}_i, \bar{y}_k)$ represents the value of the function at some point of the corresponding rectangle. Now take the limit of this sum as l and m increase indefinitely, while at the same time the dimensions of the rectangles decrease indefinitely. We then define the simultaneous integral as the limit thus obtained. The simultaneous integral is therefore equivalent to a simultaneous limit.

In the following pages we shall consider simultaneous integrals as thus defined. We must now answer the following question. If the simultaneous integral $\int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ exists, what is the necessary and sufficient condition

that the simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ shall also exist? This condition is given in the following theorem.

Theorem 1.*—If (1) $S_n(x, y) = \sum_{k=1}^{n} u_k(x, y)$, for all values x and y of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b; \tag{A}$$

(2) $\lim_{n=\infty} S_n(x, y) = S(x, y)$ for every constant pair of values (x, y) of region (A), where both $S_n(x, y)$ and $S(x, y)$ are finite;

(3) the simultaneous integral $\int_a^\beta \int_a^b S_n(x, y)\, dxdy$ exists for every constant value of n;

then the necessary and sufficient condition that the simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ shall exist is that, having chosen an arbitrarily small $\varepsilon > 0$ and $\sigma > 0$, independent of each other, and having also chosen an $n = m_1$ we can then always find an $m_2 > m_1$, such that for all values (x, y) of region (A), excepting at most for those contained within a set of rectangles finite in number and of total area less than ε, and for some number m which may vary with (x, y) but where always $m_1 < m < m_2$, we have

$$|R_m(x, y)| \equiv |S(x, y) - S_m(x, y)| < \sigma.$$

This condition may be expressed otherwise as follows: Choose as coordinate axes the axes X, Y, and $1/n$ and choose ε and σ as above. We must then be able to find a plane $1/n = m_s$ and between this plane and the plane $1/n = 0$ a finite number of planes, $1/n = m_1, m_2, m_3, \ldots, m_p$, such that after having cut out from these within the given region a finite number of rectangles of total area less than ε, these rectangles to have their sides parallel to the coordinate axes X and Y, the function $S_n(x, y)$ shall converge uniformly by areas to $S(x, y)$ throughout the rest of the region (A). It follows at once from theorem 4 of section 1 that $S(x, y)$ must be continuous in (x, y) together throughout the rest of the region. If the above conditions are fulfilled, then we say with Arzela that $S_n(x, y)$ "converges uniformly by areas in general" to $S(x, y)$.

The condition of the theorem is necessary. We suppose then that the

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*See Arzela, "Sulle serie di funzioni". R. Accademia dell 'Instituto di Bologna, 1900.

simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ exists over the region. The set of points at which $S(x, y)$ has an (x, y) discontinuity greater than or equal to σ, can therefore be at most a discrete set, that is to say, these points must be such that they can be enclosed in a set of rectangles finite in number and of total area less than ε. Let us denote this set by $t_1^{(0)}, t_2^{(0)}, t_3^{(0)}, \ldots, t_p^{(0)}$. Since now the simultaneous integral $\int_a^\beta \int_a^b S_n(x, y)\, dxdy$ also exists on each of the planes $1/n = m_1, m_2, \ldots$ the points at which $S_{m_k}(x, y)$, $(k = 1, 2, 3, \ldots)$, has an (xy) discontinuity greater than or equal to σ can form at most a discrete set, and hence

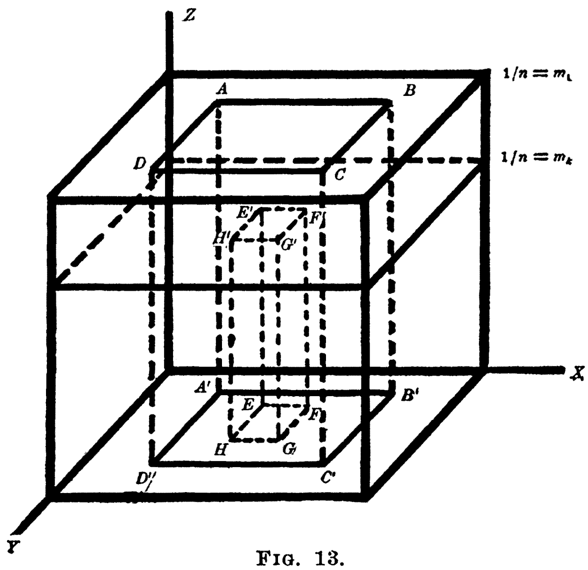

FIG. 13.

on each of the planes $1/n = m_1, m_2, m_3, \ldots$, this set of points can be enclosed in a finite set of rectangles of total area less than ε. Denote this set of rectangles by $t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, \ldots, t_q^{(k)}$, for any plane $1/n = m_{m_k}$. Now let $ABCD$ be one of the rectangles $t^{(1)}$ in the plane $1/n = m_1$ and let its projection on the plane $1/n = 0$ be $A'B'C'D'$. See figure 13.

We can now assert that if $A'B'C'D'$ does not contain any of the rectangles $t^{(0)}$ or a part of such a rectangle, then as we pass from $1/n = m_1$ to $1/n = m_2$, to $1/n = m_3$, etc., we shall ultimately reach a plane $1/n = m_k$, such that the projections of $ABCD$ on $1/n = m_k$ and on all planes between this and the plane $1/n = 0$, do not contain a rectangle $t^{(k)}$ or a part of such a rectangle. In other words, we can find a number $1/n = m_k$ such that for this and all values between

this and $1/n = 0$ the function $S_k(x, y)$ is continuous in (x, y) together throughout the corresponding rectangle. Suppose namely that this is not the case. Then as we pass through the values $1/n = m_1, m_2, m_3, \ldots,$ we always have a rectangle in every plane or at least in a set of planes dense at $1/n = 0$, such that this rectangle lies within the projection of $ABCD$ on the plane in question and furthermore such that within this rectangle the function $S_n(x, y)$ has (x, y) discontinuities greater than or equal to σ. Let us denote the set of rectangles we thus obtain by $T_1, T_2, T_3, \ldots, T_k, \ldots$ Two cases may now arise: either T_k has a lower limit T greater than zero or it has the lower limit zero as k increases indefinitely.

In the first case let $T = EFGH$ (see figure 13). Then no matter how large we take k, we always have in the plane $1/n = m_k$ a rectangle $E'F'G'H'$, corresponding to $EFGH$, such that within this rectangle the function $S_n(x, y)$ has (x, y) discontinuities greater than or equal to σ. Let (x_0, y_0) be a point of this rectangle at which the (x, y) discontinuity is greater than or equal to σ. Then we have for all values of n and for some point (x, y) of this rectangle

$$|S_n(x, y) - S_n(x_0, y_0)| > \sigma. \tag{1}$$

Since by hypothesis $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, we have also

$$|S(x, y) - S_n(x, y)| < 1/4\,\sigma; \quad n > m_k. \tag{2}$$

Furthermore since $S(x, y)$ is continuous in (x, y) together throughout the rectangle $EFGH$, we have

$$|S(x_0, y_0) - S(x, y)| < 1/4\,\sigma. \tag{3}$$

Combining these three inequalities we obtain

$$|S_n(x_0, y_0) - S(x_0, y_0)| > 1/4\,\sigma; \quad n > m_k. \tag{4}$$

Inequality 4 however contradicts the hypothesis that $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every pair of values (x, y) of region (A), and hence the assumption we made is false. It therefore follows that, if we have in each of a set of planes dense at $1/n = 0$ a rectangle of lower limit $T \neq 0$, within which the function $S_n(x, y)$ has an (x, y) discontinuity greater than or equal to σ, then the function $S(x, y)$ has such an (x, y) discontinuity within the rectangle T in the plane $1/n = 0$.

Suppose in the second place that the limit of T is zero. We have then in the plane $1/n = m_s$ a rectangle T_s within which any (x, y) discontinuity of

$S_n(x, y)$ is greater than or equal to σ. Furthermore as s increases indefinitely the area of this rectangle approaches zero. Let this occur at the point (x', y') and let (x'', y'') be any other point of the rectangle T_s. If now we drop a perpendicular from (x'', y'') to the plane $1/n = 0$, then one of two things will happen; either there exists a plane $1/n = m_k$, such that the perpendicular does not meet a rectangle $t^{(i)}$ in any plane $1/n = m_i$ between $1/n = m_k$ and $1/n = 0$, or no matter what plane $1/n = m_k$ we choose, the perpendicular will always meet a rectangle $t^{(k)}$. We must now show that in the latter case the perpendicular also meets one of the rectangles $t^{(0)}$ in the plane $1/n = 0$. We assume that the perpendicular meets a rectangle $t^{(k)}$ in every plane $1/n = m_k$. Then for every point of this rectangle

$$|S_{m_k}(x, y) - S_{m_k}(x'', y'')| > \sigma. \tag{1}$$

Since moreover $\lim_{n=\infty} S_n(x, y) = S(x, y)$ for every point of region (A), it follows that

$$|S(x, y) - S_{m_k}(x, y)| < 1/4\, \sigma \tag{2}$$

for every m_k greater than some m'. Likewise

$$|S_{m_k}(x'', y'') - S(x'', y'')| < 1/4\, \sigma \tag{3}$$

for every m_k greater than some m''. Let $m'' > m'$; then for every m_k greater than m'', (1), (2), and (3), may be added and hence

$$|S(x, y) - S(x'', y'')| > 1/4\, \sigma \tag{4}$$

for every (x, y) of some rectangle about (x'', y'') in the plane $1/n = 0$. Hence $S(x, y)$ is discontinuous in (x, y) together at (x'', y'') and the perpendicular from (x'', y'') therefore must meet one of the rectangles $t^{(0)}$ in the plane $1/n = 0$. In a like manner we can prove that the perpendicular from (x', y') meets one of the rectangles $t^{(0)}$ in the plane $1/n = 0$.

We are now ready to complete the proof of the original theorem. In order to do so, let us cut out from the plane $1/n = 0$ all the rectangles $t^{(0)}$, that contain the points at which $S(x, y)$ has an (x, y) discontinuity greater than or equal to σ. As has been shown, this set of rectangles is finite in number and of total area less than ε. Lines through the vertices of these rectangles parallel to the X and Y axes divide the remaining part of region (A) into a finite number of rectangles. Number these 1, 2, 3, ..., q. Now consider rectangle number 1. From what has been proved it follows that there must exist a plane $1/n = m_k$ such that for this plane and for all planes between this and the plane $1/n = 0$

the function $S_n(x, y)$ is continuous in (x, y) together throughout the rectangle in question. Hence by theorem 4 of section 10 the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout this rectangle. In a like manner we prove that the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout each of the remaining rectangles $2, 3, \ldots, q$. We have thus shown that, after we have cut out from region (A) a finite number of rectangles of total area less than ε, the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout the rest of region (A). This proves that the condition of the theorem is necessary.

The condition is also sufficient. By hypothesis then $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout region (A), excepting at the points of a finite number of rectangles of total area less than ε. We can therefore choose at pleasure a plane $1/n = m_s$ and there will then exist between this plane and the plane $1/n = 0$ a finite number of planes having the following properties. From these planes, including the plane $1/n = 0$, are cut out a finite number of rectangles, $t_1, t_2, t_3, \ldots, t_p$, of total area less than ε. These rectangles must be such that, from some m_s on, their projections on the plane $1/n = 0$ cannot completely fill region (A), otherwise the function $S(x, y)$ would be totally discontinuous in (x, y) together throughout region (A). In the remaining part of each plane there exists a finite number of adjacent rectangles (each of area greater than $G > 0$) whose projections on the plane $1/n = 0$ completely fill the remaining part of region (A), such that for every point (x, y) of these rectangles

$$|S_k(x, y) - S(x, y)| < \sigma. \tag{1}$$

Let us denote $S_k(x, y)$ for the points of these rectangles by $K(x, y)$. Then the following inequality always is true

$$|K(x, y) - S(x, y)| < \sigma. \tag{2}$$

Moreover the simultaneous integral $\int_\alpha^\beta \int_a^b K(x, y)\, dxdy$ exists throughout the region for which $K(x, y)$ is defined. Now consider the identity

$$S(x, y) \equiv K(x, y) + S(x, y) - K(x, y),$$

which holds for all points (x, y) for which $K(x, y)$ is defined. In the right member of this identity $K(x, y)$ may have an (x, y) discontinuity greater than or equal to σ only at a discrete set of points. The function $[S(x, y) - K(x, y)]$ can have such an (x, y) discontinuity only at points of the rectangles

$$t_1, t_2, t_3, \ldots, t_p.$$

It follows from this that the sum $K(x, y) + S(x, y) - K(x, y)$ can have such an (x, y) discontinuity only at a discrete set of points. This being the case, it is at once evident that the simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ exists.

The theorem just proved holds also in the general case of a function of three variables, the proof being similar to that just given.

Having thus obtained the necessary and sufficient condition that the simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ exists, the next problem is the following.

If the simultaneous integral

$$\int_a^\beta \int_a^b \lim_{n=\infty} S_n(x, y)\, dxdy = \int_a^\beta \int_a^b S(x, y)\, dxdy = \theta(x, y)$$

exists for every point (x, y) of region (A) and if it is finite and continuous, under what conditions will the simultaneous integral $\lim\limits_{n=\infty} \int_a^\beta \int_a^b S_n(x, y)\, dxdy$ exist and equal $\theta(x, y)$?

We have here two cases to consider. For every fixed value of n but for a variable pair of values (x, y) the function $|S_n(x, y)|$ has an upper limit $G > 0$. This G is a function of n, if we consider n as variable. Then we have the following two cases:

(1) G has an upper finite limit G' as n increases indefinitely;

(2) G has no finite upper limit as n increases indefinitely.

These two cases give rise to different conditions which are stated in the following theorems. Arzela* has proved the analogous theorems for a series of functions of a single variable. The proofs given below are generalizations of Arzela's proofs.

Theorem 2.—If $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$ where each of the simultaneous integrals $\int_a^\beta \int_a^b u_k(x, y)\, dxdy$ exists throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if also $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every (x, y) of region (A), and if for every constant value of n and for every (x, y) of region (A) we have $S_n(x, y) < G$,

* Cf. Arzela: "Sulle serie di funzioni"; R. Accademia dell' Instituto di Bologna, 1900.

where G is finite, then the necessary and sufficient condition that the simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ shall exist at every point of region (A) and that at the same time

$$\int_a^\beta \int_a^b S(x, y)\, dxdy = \lim_{n=\infty} \int_a^\beta \int_a^b S_n(x, y)\, dxdy$$

is that $S(x, y)$ shall converge uniformly by areas in general to $S(x, y)$ throughout region (A).

That the condition is necessary follows immediately from theorem 1 of this section.

The condition is also sufficient. By hypothesis therefore $S_n(x, y) < G$ for every constant n and for every (x, y) of region (A). Hence, since $\lim_{n=\infty} S_n(x, y) = S(x, y)$

$$|S(x, y) - S_n(x, y)| < 2G \tag{1}$$

so that $S(x, y)$ also has a finite upper limit. Let us now choose independently of each other an arbitrarily small $\sigma < 0$ and $\varepsilon < 0$. Since $S_n(x, y)$ converges uniformly by areas in general to $S(x, y)$ throughout region (A), we can find an s such that the points at which

$$|S(x, y) - S_{s+p}(x, y)| \geqq \sigma, \quad (p = 1, 2, 3, \ldots) \tag{2}$$

form a discrete set, that is they can be enclosed in a finite number of rectangles of total area less than ε. Consider now the function $[S(x, y) - S_{s+p}(x, y)]$. This function can evidently have an (x, y) discontinuity greater than or equal to $1/2\sigma$ only at points (x, y) for which (2) holds, that is at a discrete set of points, and this is true for all values of p. Hence the simultaneous integral $\int_a^\beta \int_a^b [S_{s+p}(x, y) - S(x, y)]\, dxdy$ exists. Moreover the value of this integral is less than or at most equal to $(x - a)(y - a)\sigma + 2G\varepsilon$. Since by hypothesis the simultaneous integrals $\int_a^\beta \int_a^b S_{s+p}(x, y)\, dxdy$ and $\int_a^\beta \int_a^b S(x, y)\, dxdy$ also exist for any (x, y) of the region, we have

$$\int_a^\beta \int_a^b S_{s+p}(x, y)\, dxdy - \int_a^\beta \int_a^b S(x, y)\, dxdy = \int_a^\beta \int_a^b [S_{s+p}(x, y) - S(x, y)]\, dxdy < (x - a)(y - a)\sigma + 2G\varepsilon < \varepsilon'.$$

Hence we obtain in the limit $\lim_{n=\infty} \int_a^\beta \int_a^b S_n(x, y)\, dxdy = \int_a^\beta \int_a^b S(x, y)\, dxdy$, which was to be proved.

Before taking up the case where G has no upper finite limit, we must first prove a couple of lemmas. These theorems were first proved by Scheeffer* for a function of single variable.

Lemma 1.—If $F(x, y)$ and $f(x, y)$ are two functions of x and y which are continuous in (x, y) together throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

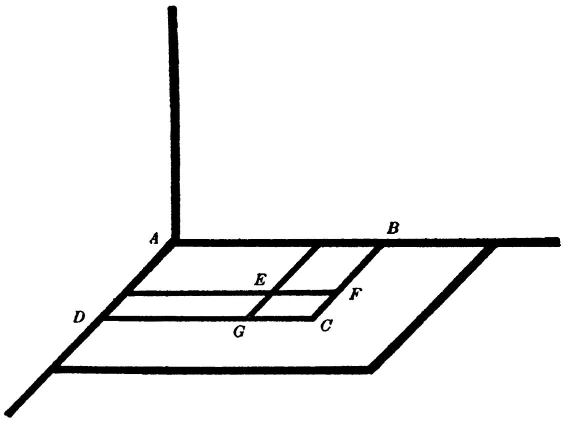

FIG. 14.

and if for every point (x, y) of this region two corresponding difference quotients are always equal, that is if

$$\frac{F(x+h, y+k) - F(x, y+k) - F(x+h, y) + F(x, y)}{hk} =$$

$$\frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk},$$

these values being finite, then $F(x, y)$ and $f(x, y)$ can differ at most by a constant quantity.

In order to prove this, we consider the function

$$\psi(x, y) = c(x-\alpha)(y-a) + F(x, y) - f(x, y) - [F(\alpha, a) - f(\alpha, a)]. \tag{1}$$

Let $DF(x, y)$, $Df(x, y)$, and $D\psi(x, y)$, be the corresponding difference quotients and let c be any positive quantity. Then $\psi(x, y)$ can never be negative in the region (A). In order to prove this we assume the contrary and suppose that $\psi(x, y)$ can be negative at some point $(\bar{x}, \bar{y})$ of the given region. Let this be the point C and consider now the rectangle $A(\alpha, a)$, $B(\bar{x}, a)$, $C(\bar{x}, \bar{y})$, $D(\alpha, \bar{y})$, see figure 14.

* Cf. L. Scheeffer: Acta Mathematica, Vol. 5 (1884), pages 84 and 183.

Within this rectangle the points (x', y') at which $\psi(x, y)$ is not negative are such that the absolute value of $\sqrt{[x'^2 + y'^2]}$ has an upper limit which is finite. Suppose that this occurs at the point $(\bar{x}', \bar{y}') = E$. Since $\psi(x, y)$ is continuous at every point of region (A) in (x, y) together, it follows that $\psi(\bar{x}', \bar{y}') = 0$. But $\bar{x}' < \bar{x}$ and $\bar{y}' < \bar{y}$. Moreover we have for every point of the rectangle *EFCG*

$$\psi(x, y) - \psi(\bar{x}', \bar{y}') = 0.$$

From this follows that $D\psi(\bar{x}', \bar{y}') = 0$. On the other hand, $Df(x, y)$ and $D\psi(x, y)$ are both finite, and hence we have from (1) that $D\psi(x, y) = c$ for every point (x, y) of region (A). This last statement is surely true for

$$D\psi(x, y) = Dc(x - \alpha)(y - a) + DF(x, y) - Df(x, y) = Dc(x - \alpha)(y - a).$$

But $Dc(x - \alpha)(y - a) = c$. We are thus led to a contradiction, and since this holds for every positive value of c, it follows that $\psi(x, y)$ can never be negative. Hence

$$F(x, y) - f(x, y) - [F(\alpha, a) - f(\alpha, a)]$$

cannot ever be negative. In a like manner we can prove that this difference cannot ever be positive and hence it must be zero. This proves the lemma.

This lemma may be extended as follows:

Lemma 2.—If the conditions of lemma 1 hold, with the exception that the two difference quotients are not equal to each other at an enumerable set of points, that is, are not everywhere dense in any sub-region of region (A), then the two given functions can differ at most by a constant quantity.

Let L be a set of exceptional points, and suppose that L is not everywhere dense in any sub-region of region (A). We then have a set D of sub-regions d which do not contain any of the exceptional points, and we can thus determine a closed set of points C, with respect to which L is everywhere dense. In every sub-region d the preceding theorem holds, so that for every sub-region d' contained wholly within d the difference $F(x, y) - f(x, y)$ is constant. But this difference is continuous in (x, y) together and hence it follows that it is constant also in the sub-region d. If this is so, then $F(x, y) - f(x, y)$ is either a constant or it is "streckenweise" discontinuous. (See Schoenfliess: Punktmannigfaltigkeiten, p. 166). If L is enumerable, then the above difference is constant, while if L has the "Mächtigkeit" of the continuum then this difference is "streckenweise" discontinuous. This proves the lemma.

We are now ready to take up our final theorem.

Theorem 3.—If we have

$$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y),$$

where $\lim_{n=\infty} S_n(x, y) = S(x, y)$ for every (x, y) of

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if $S(x, y)$ is completely defined and is always less in absolute value than some finite quantity, and if both the simultaneous integrals $\int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ and $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exist, then the sufficient condition that

$$\lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy$$

where $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ is finite, is that the set of points (x', y') for which we cannot assign a circle of radius $r > 0$ and an integer m, such that for every point (x, y) of this circle and for every $n > m$,

$$| S_n(x, y) | < G \qquad (G \text{ finite})$$

is enumerable.

In this case we have in the plane $1/n = z$ certain points (x', y') in whose neighborhood $S_n(x, y)$ does not have a finite upper limit. About such a point as center we can therefore construct a circle of radius $r > 0$, and at the same time find a plane $z = 1/m_s$, such that for every point (x, y) of this circle and for every value of $p = 1, 2, 3, \dots$,

$$| S_{m_s+p}(x, y) | > G. \qquad (G \text{ finite})$$

This set of points (x', y') must evidently be a closed set, for it is a set for which the limit of the oscillation k is greater than some fixed finite number. We have now by hypothesis

$$\int_\alpha^x \int_a^y S(x, y)\, dxdy = \theta(x, y)$$

and

$$\lim_{n=\infty} \int_\alpha^x \int_a^y S_n(x, y)\, dxdy = \phi_0(x, y).$$

Now let (x, y) be some point not belonging to the set (x', y'). There exists then about it a circle of radius $r > 0$ for which

$$| S_{m_s+p}(x, y) | < G.$$

Consider now the ratios

$$\frac{\theta(x+h, y+k) - \theta(x+h, y) - \theta(x, y+k) + \theta(x, y)}{hk}$$

and

$$\frac{\phi_0(x+h, y+k) - \phi_0(x+h, y) - \phi_0(x, y+k) + \phi_0(x, y)}{hk}$$

where $(x+h, y+k)$ is to be a point of the above circle. Choose an arbitrarily small $\lambda > 0$; we can then find a plane $z = 1/m_s$, such that for every value of $p = 1, 2, 3, \ldots.$

$$\frac{\phi_0(x+h, y+k) - \phi_0(x+h, y) - \phi_0(x, y+k) + \phi_0(x, y)}{hk} =$$

$$\frac{\phi_{m_s+p}(x+h, y+k) - \phi_{m_s+p}(x+h, y) - \phi_{m_s+p}(x, y+k) + \phi_{m_s+p}(x, y)}{hk} +$$

$$-1 < \mu < 1,$$

since $\lim\limits_{n=\infty} \phi_{m_s+p}(x, y) = \phi_0(x, y)$ for every point (x, y). But we have by hypothesis

$$\frac{\phi_{m_s+p}(x+h, y+k) - \phi_{m_s+p}(x+h, y) - \phi_{m_s+p}(x, y+k) + \phi_{m_s+p}(x, y)}{hk} =$$

$$1/hk \int_x^{x+h} \int_y^{y+k} S(x, y)\, dxdy + \mu\lambda$$

$$-1 < \mu < 1,$$

where λ is independent of x, y, h, and k. Since now $|S_n(x, y)| < G$ at the points in question, it follows from theorem 2 that

$$1/hk \int_x^{x+h} \int_y^{y+k} S_{m_s+p}(x, y)\, dxdy = 1/hk \int_x^{x+h} \int_y^{y+k} S(x, y)\, dxdy + \mu' \frac{\lambda}{h}$$

$$-1 < \mu' < 1.$$

Hence

$$\frac{\phi_0(x+h, y+k) - \phi_0(x+h, y) - \phi_0(x, y+k) + \phi_0(x, y)}{hk} =$$

$$1/hk \int_x^{x+h} \int_y^{y+k} S(x, y)\, dxdy + \mu' \frac{\lambda}{h} + \mu\lambda$$

$$= \frac{\theta(x+h, y+k) - \theta(x+h, y) - \theta(x, y+k) + \theta(x, y)}{hk} + \lambda \left\{ \mu + \frac{\mu'}{h} \right\}.$$

which holds for every x, $x+h$, y, and $y+k$, determined as above. We thus obtain two different quotients which are equal to each other at all points of region (A), excepting at a closed enumerable set of points, and hence by lemma 2 they can differ from each other at most by a constant quantity. Hence

$$\phi_0(x, y) = \theta(x, y) + \text{constant}.$$

But

$$\phi_0(x, y) = \theta(\alpha, y) = 0$$

hence

$$\phi_0(x, y) = \theta(x, y),$$

which was to be proved.

By a proof similar to that just employed, we can show that the theorem holds also when we replace the function $S_n(x, y)$ by the general function $f(x, y, z)$.

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